J. Fluid Mech. (1972), *vol.* 54, *part* 1, *pp.* 113–127
Printed in Great Britain

Linear spatial stability of pipe Poiseuille flow

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(Received 25 November 1971)

A theoretical study of the spatial stability of Poiseuille flow in a rigid pipe to infinitesimal disturbances is presented. Both axisymmetric and non-axisymmetric disturbances are considered. The coupled, linear, ordinary differential equations governing the propagation of a disturbance that has a constant frequency and is imposed at a specified location in the fluid are solved numerically for the complex eigenvalues, or wavenumbers, each of which defines a mode of propagation. A series solution for small values of the pipe radius is derived and step-by-step integration to the pipe wall is then performed. In order to ascertain the number of eigenvalues within a closed region, an eigenvalue search technique is used. Results are obtained for Reynolds numbers up to 10000. For these Reynolds numbers it is found that the pipe Poiseuille flow is spatially stable to all infinitesimal disturbances.

1. Introduction

The transition from laminar to turbulent flow of a viscous fluid in a pipe was first observed experimentally by Reynolds (1883). Subsequently, in an effort to explain the phenomenon of transition, numerous studies of the stability of fluid flow have been carried out. For viscous flow through a circular pipe, the theoretical studies of temporal stability by Sexl (1927*a*, *b*), Pretsch (1941), Pekeris (1948), Corcos & Sellars (1959), Lessen, Sadler & Liu (1968) and Burridge (1972) have all demonstrated stability to infinitesimal axisymmetric and non-axisymmetric disturbances that are applied at an initial instant everywhere in the fluid and grow or decay with time. For the physically more realistic problem of spatial stability where a disturbance, imposed at a specific location in the fluid, grows or decays with axial distance, both experimental (Leite 1959) and theoretical (Gill 1965) studies have also shown stability. However, these latter investigations were carried out for axisymmetric disturbances only. For the non-axisymmetric disturbance, experiments (Lessen, Fox, Bhat & Liu 1964; Fox, Lessen & Bhat 1968) indicate spatial instability but prior to this paper no theoretical analysis has been carried out. The present paper provides results of a theoretical analysis of the spatial stability of pipe Poiseuille flow to both axisymmetric and non-axisymmetric infinitesimal disturbances.

2. Stability equations and boundary conditions

In the theory of the stability of laminar flows the motion is decomposed into a mean flow (whose stability constitutes the subject of the investigation) and a disturbance superimposed on it. Then, starting with the continuity and Navier–Stokes equations for an incompressible Newtonian fluid, a set of linear partial differential equations is obtained by neglecting the product of disturbance velocity components with themselves and their spatial derivatives. After this set has been made dimensionless each component, say $\phi(r,\theta,z,t)$, of the Fourier series corresponding to the arbitrary disturbance is assumed to be of the form

$$\phi(r,\theta,z,t) = \mathrm{Re}\,[\hat{\phi}(r,\theta,z,t)] = \mathrm{Re}\,[\bar{\phi}(r)\exp(kz+in\theta-i\omega t)],$$

where Re denotes the real part of the complex functions $\hat{\phi}$ and $\bar{\phi}$, r,θ,z and t are the dimensionless radial, angular, axial and time co-ordinates, ω is the real frequency, n the angular wavenumber and k the axial complex wavenumber. Substitution of such expressions for the disturbance pressure and velocity components into the original linear equations leads to the following set of four coupled, linear, ordinary differential equations:

$$\left.\begin{aligned}
&\left(D+\frac{1}{r}\right)\bar{v}_r+\frac{in}{r}\bar{v}_\theta+k\bar{v}_z = 0,\\
&\left[D^2+\frac{1}{r}D-\left\{\frac{n^2+1}{r^2}-k^2+R(kV_z-i\omega)\right\}\right]\bar{v}_r-i\frac{2n}{r^2}\bar{v}_\theta-RD\bar{p} = 0,\\
&\left[D^2+\frac{1}{r}D-\left\{\frac{n^2+1}{r^2}-k^2+R(kV_z-i\omega)\right\}\right]\bar{v}_\theta+i\frac{2n}{r^2}\bar{v}_r-\frac{in}{r}R\bar{p} = 0,\\
&\left[D^2+\frac{1}{r}D-\left\{\frac{n^2}{r^2}-k^2+R(kV_z-i\omega)\right\}\right]\bar{v}_z-R\bar{v}_rDV_z-Rk\bar{p} = 0,
\end{aligned}\right\}\quad (2.1)$$

where D is the operator d/dr, R is the Reynolds number of flow, $\bar{v}_r$, $\bar{v}_\theta$, $\bar{v}_z$ and $\bar{p}$ are the dimensionless complex eigenfunctions for disturbance velocity components and pressure, and V_z is the dimensionless mean velocity of flow given, for pipe Poiseuille flow, by $V_z = (1-r^2)$. $R = 2r_p\mathscr{V}/\nu$, where r_p is the radius of the pipe, $\mathscr{V}$ is the average velocity and ν is the kinematic viscosity of the fluid.

The physical restrictions at the centre of the pipe require that fluid velocity and pressure be bounded and continuous at $r = 0$ (and at any other r for that matter). Therefore, following Batchelor & Gill (1962), the boundary conditions at $r = 0$ are

$$\left.\begin{aligned}
\bar{v}_z(0) = \bar{p}(0) = 0 \quad &\text{for} \quad n \neq 0,\\
\bar{v}_z(0)\ \text{finite},\quad \bar{p}(0)\ \text{finite} \quad &\text{for} \quad n = 0.
\end{aligned}\right\}\quad (2.2)$$

$$\left.\begin{aligned}
\bar{v}_r(0) = \bar{v}_\theta(0) = 0 \quad &\text{for} \quad n \neq 1,\\
\bar{v}_r(0)+i\bar{v}_\theta(0) = 0 \quad &\text{for} \quad n = 1.
\end{aligned}\right\}\quad (2.3)$$

The physical restrictions at $r = 1$ for the rigid impermeable pipe wall, assuming no slip, are

$$\bar{v}_r(1) = \bar{v}_\theta(1) = \bar{v}_z(1) = 0 \quad \text{for all } n. \quad (2.4)$$

With these boundary conditions the determination of the dimensionless complex wavenumber k for a given frequency ω and Reynolds number R with n as a parameter is an eigenvalue problem. The flow is considered to be unstable when the disturbance grows with z and vice versa. In terms of k, stability is implied for $k_r < 0$ and instability for $k_r > 0$, where k_r is the real part of k, k_i being the imaginary part.

3. Solution of disturbance equations

The four coupled differential equations (2.1) are too complex to be amenable to analytical solution. Even for an axisymmetric disturbance ($n = 0$), for which the set of four equations reduces to a set of three, an earlier analytical solution (Gill 1965) was mainly successful for only two particular types of disturbance: one confined to a region close to the wall and the other close to the centre of the pipe. The numerical method developed here is not limited by any such restrictions and can be extended to compressible flow in an elastic tube.

Briefly, the eigenfunctions are expanded as a power series in r for r small. The series solution is carried up to a small but finite value of r because of practical difficulties in summing a very large number of terms. The solution can then be continued by any of the standard step-by-step integration techniques (Runge–Kutta or a predictor–corrector method) to the pipe wall. The whole procedure is iterated until the boundary conditions at the wall are satisfied, thus resulting in a value for the complex wavenumber k.

For a non-axisymmetric disturbance, it is simpler to develop the series expansions of the eigenfunctions $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$ in terms of the combinations

$$\left.\begin{aligned} \bar{f}(r) &= \bar{v}_r(r) + i\bar{v}_\theta(r), \\ \bar{g}(r) &= \bar{v}_r(r) - i\bar{v}_\theta(r). \end{aligned}\right\} \tag{3.1}$$

Once $\bar{f}(r)$ and $\bar{g}(r)$ are known, it is simple to calculate $\bar{v}_r(r)$ and $\bar{v}_\theta(r)$. The transformed form of (2.1) in terms of $\bar{f}, \bar{g}, \bar{v}_z$ and $\bar{p}$ is easily obtained and the boundary conditions on $\bar{f}$ and $\bar{g}$ follow directly from (2.2)–(2.4).

Let $\mathbf{S}$ be a vector such that $\mathbf{S} \equiv (S_1, S_2, S_3, S_4) \equiv \{\bar{f}, \bar{g}, \bar{v}_z, \bar{p}\}$. The series expansion for each eigenfunction is then assumed to be of the form

$$S_j(r) = r^{a_j}(S_{1j} + S_{2j}r + S_{3j}r^2 + \ldots + S_{lj}r^{l-1} + \ldots) \quad (j = 1, 2, 3, 4), \tag{3.2}$$

where the S_{ij} are complex constants with

$$(S_{i1}, S_{i2}, S_{i3}, S_{i4}) \equiv (F_i, G_i, U_i, P_i). \tag{3.3}$$

When (3.2) is substituted into the transformed form of (2.1) and the coefficients of r^α in the resulting equations are set to zero for all α, the values of a_j that satisfy the boundary conditions at $r = 0$ are found to be

$$\{a_1, a_2, a_3, a_4\} = \{(n+1), (n-1), n, n\}. \tag{3.4}$$

It is also found that $F_i = G_i = U_i = P_i = 0$, if i is even. Thus, by redefining the S_{ij} we can rewrite the series expansion (3.2) as

$$S_j = r^{a_j}(S_{1j} + S_{2j}r^2 + S_{3j}r^4 + \ldots + S_{lj}r^{2(l-1)} + \ldots) \quad (j = 1, 2, 3, 4). \tag{3.5}$$

From (3.5), the recurrence relations for the F_i, G_i, U_i and P_i are (Garg 1971)

$$\left.\begin{aligned}
G_{l+1} &= \frac{1}{2l}\left[RP_l - \frac{1}{2(n+l-1)}\sum_{j=1}^{m} G_{l+1-j}B_j\right],\\
U_{l+1} &= \frac{1}{4l(n+l)}\left[kRP_l - \sum_{j=1}^{m} U_{l+1-j}B_j + R\sum_{j=2}^{m}(j-1)C_j(F_{l+1-j}+G_{l+2-j})\right],\\
P_{l+1} &= \frac{1}{R}\left[-kU_{l+1}+\frac{1}{4}\left\{\frac{1}{l}\sum_{j=1}^{m}F_{l+1-j}B_j+\frac{1}{(n+l)}\sum_{j=1}^{m}G_{l+2-j}B_j\right\}\right],\\
F_{l+1} &= \frac{1}{2(n+l+1)}\left[RP_{l-1}+\frac{1}{2l}\sum_{j=1}^{m}F_{l+1-j}B_j\right],
\end{aligned}\right\} \tag{3.6}$$

where any term having a factor with either a zero or negative subscript is set to zero, and

$$\left.\begin{aligned}
B_1 &= k^2 - R(kC_1 - i\omega),\\
B_j &= -kRC_j \quad (j = 2, 3, \ldots),\\
l &= 1, 2, 3, \ldots,\\
C_j &= \text{coefficients in the series expansion of } V_z,
\end{aligned}\right\} \tag{3.7}$$

viz.

$$V_z(r) = C_1 + C_2r^2 + C_3r^4 + \ldots + C_m r^{2(m-1)}, \tag{3.8}$$

so that for Poiseuille flow in a pipe $C_1 = -C_2 = 1$, and the higher C's are all zero.

Only three of the first four constants F_1, G_1, U_1 and P_1 in the series expansions are independent owing to the relation

$$F_1 = \frac{1}{n+1}\left(\frac{B_1}{4n}G_1 - \frac{R}{2}P_1 - kU_1\right). \tag{3.9}$$

Taking G_1, U_1 and P_1 to be independent, all the F_l, G_l, U_l and P_l with $l > 1$ can be expressed in terms of G_1, U_1 and P_1 with the help of (3.9) and recurrence relations (3.6). This enables any eigenfunction to be expressed as a sum of three terms; for example, the axial velocity eigenfunction $\bar{v}_z(r)$ may be written as

$$\bar{v}_z(r) = v_{z1}(r)G_1 + v_{z2}(r)U_1 + v_{z3}(r)P_1. \tag{3.10}$$

This results in three independent sets of solutions $(v_{r1}, v_{\theta1}, v_{z1}, p_1)$, $(v_{r2}, v_{\theta2}, v_{z2}, p_2)$ and $(v_{r3}, v_{\theta3}, v_{z3}, p_3)$. The set of equations (2.1) is thus equivalent to three sets since each of the three sets of solutions must satisfy (2.1) independently. Starting from the small value of r up to which the series solution is carried, each of the three sets of (2.1) is solved independently by a step-by-step integration technique to the pipe wall. It is the boundary conditions at the rigid pipe wall that preclude all but a certain combination of the three independent solutions for every eigenfunction. These boundary conditions require that the disturbance velocity components be zero at $r = 1$, so that in view of (3.10) we must have

$$\begin{bmatrix} v_{r1}(1) & v_{r2}(1) & v_{r3}(1)\\ v_{\theta1}(1) & v_{\theta2}(1) & v_{\theta3}(1)\\ v_{z1}(1) & v_{z2}(1) & v_{z3}(1)\end{bmatrix}\begin{bmatrix} G_1\\ U_1\\ P_1\end{bmatrix} = \mathbf{0}. \tag{3.11}$$

For a non-trivial solution, then, the determinant d of the coefficient matrix in (3.11) must vanish or, since d is in general complex, its absolute value must be zero, that is,

$$|d| = 0. \tag{3.12}$$

Further, since the elements of the determinant d are functions of the complex wavenumber k, frequency ω and Reynolds number R, only certain combinations of these parameters will allow (3.12) to be satisfied, thus yielding the eigenvalue k.

For an axisymmetric disturbance ($n = 0$), the circumferential component of the disturbance velocity ($\bar{v}_\theta(r)$) can be arbitrarily assumed to vanish, thus providing an obvious simplicity which is lacking in the case of a non-axisymmetric disturbance. A more important consequence of this simplicity is that for an axisymmetric disturbance each eigenfunction can be represented as a combination of only two linearly independent solutions, which are linked by the boundary conditions at the rigid pipe wall. In contrast to a 3×3 determinant for a non-axisymmetric disturbance, therefore, a determinant which is only 2×2 determines the eigenvalue for an axisymmetric disturbance. The analysis is thus similar to that given above.

4. Eigenvalue search technique

This investigation, being mainly concerned with locating possible spatial instabilities of Poiseuille flow in a pipe, requires that the whole (or a significant part) of the complex-k plane be explored for possible eigenvalues. This task is, unfortunately, not an easy one since (3.12), which determines the eigenvalue, is some high order polynomial in k whose exact nature is unknown. An iterative technique, if used for this purpose, has a major drawback in its tendency to bypass one eigenvalue and converge to another which is generally more stable. Thus, use of an iterative technique alone is highly inadequate. Other techniques used in this context are based on variations of the so-called grid methods which evaluate the determinant for every mesh point in the region of interest. The most severe disadvantage of any grid method, including a recent one by Scarton & Rouleau (1972), is the large amount of computer time used in evaluating the determinant at sufficient mesh points inside and on the boundaries of a closed region of the k plane.

Such complicated and expensive heuristics are obviated by a method that follows directly from the argument principle in complex-variable theory. For a function $f(z)$, analytic except for poles in the interior of a closed curve C in the z plane and continuously differentiable on C, Cauchy's theorem gives (Nehari 1969)

$$N - P = \frac{1}{2\pi i}\int_C \frac{Df(z)}{f(z)}\,dz,$$

or

$$N - P = 2M\pi i/2\pi i = M, \tag{4.1}$$

where N and P denote respectively the number of zeros and the number of poles of $f(z)$ within the closed region C (counted with their multiplicities), D is the operator d/dz and M is an integer denoting the net multiples of 2π by which the phase angle of $f(z)$ changes as z moves once around on C.

For the present problem, each element of the determinant is a function of the complex eigenvalue k. In fact, the determinant can be considered as some high order polynomial in k because of the recurrence relations for the coefficients in the series solution of the stability equations. Therefore, the determinant $d(k)$ has no poles, that is, $P \equiv 0$ in (4.1) and $N = M$. Thus the problem of finding the number of eigenvalues within a closed region of the k plane is equivalent to counting the net multiples of 2π by which the phase angle of the determinant changes as k assumes values on the closed contour in the k plane.

The method described above is most useful in establishing the number of eigenvalues inside a closed region of the k plane. Although a complicated technique for isolating a number of these eigenvalues simultaneously rather than one at a time has been developed by Delves & Lyness (1967), it was not used for the present study because the main purpose here was not to find a number of stable modes but to determine whether the pipe flow is spatially unstable to infinitesimal disturbances. To isolate an eigenvalue from the many suggested by the argument-principle method, the region is divided into smaller subregions which are investigated individually for possible eigenvalues. The process is repeated until the region containing an eigenvalue is sufficiently small for an iterative technique to converge easily to the true eigenvalue.

5. Results

As already stated, the flow would be spatially unstable if the real part of the complex wavenumber k were positive, that is, if there were eigenvalues lying in the first quadrant of the complex-k plane for waves propagating downstream. The regions in this part of the k plane were, therefore, explored for possible eigenvalues by means of the technique described in §4. For this purpose, the azimuthal wavenumber n was taken to be 0 or 1 and the Reynolds number R and frequency ω were varied so that their product ωR covered a wide range from 200 to 10000 for each n. The net change in the phase angle of the determinant when k assumed values on the boundary of a closed region was found to be zero in each case, thus establishing that no eigenvalue exists inside any of the regions examined in the first quadrant of the k plane. Owing to limitations of the computer, these regions decreased in size with increasing values of the product ωR. Some of the regions examined in the first quadrant are given in table 1. Discounting the highly unlikely possibility of an unstable mode lying beyond these regions (i.e. for values of α and β greater than those listed in table 1), it appears that the pipe Poiseuille flow is spatially stable to infinitesimal axisymmetric disturbances and the non-axisymmetric disturbance for which $n = 1$. Later, it will be shown, cf. a similar observation by Burridge (1972) for temporal stability, that the pipe flow is relatively more stable to disturbances with $n > 1$. For an axisymmetric disturbance, the result is in agreement with earlier studies (Leite 1959; Gill 1965). For a non-axisymmetric disturbance it is the first numerical confirmation of spatial stability.

n	R	ω	α	β
0	500	1·0	3·0	4·0
0	4000	0·5	3·0	3·0
0	7000	0·7	1·4	2·2
0	10000	1·0	1·0	1·2
1	500	1·0	5·0	4·0
1	2000	0·1	2·0	1·5
1	5000	0·5	0·2	2·0
1	5000	1·0	0·1	1·0

TABLE 1. Regions explored for unstable modes
$(0 \leqslant k_r \leqslant \alpha)$, $(0 \leqslant k_i \leqslant \beta)$

While finding some stable modes, it was observed that the two parameters, frequency ω and Reynolds number R, could be combined into one, the product ωR. Thus, it was found that ωR governs, in an approximate manner, the values of kR. This provides an obvious simplification by reducing the number of parameters from two to one. For a non-axisymmetric disturbance, however, the mode shapes do have a marked difference for different combinations of ω and R for which ωR is constant. Such was not found to be the case for an axisymmetric disturbance; there the eigenfunctions were almost the same for all the cases studied with this point in mind. In view of the observation that ωR is the governing factor for $k_r R$ and $k_i R$, it may be pointed out that, if R is increased and ω decreased indefinitely in order that the product ωR remains constant, the mode will become less and less stable since k_r (and also k_i) will approach zero. Thus if an infinitesimal disturbance of a very low frequency is applied to the pipe Poiseuille flow at very high Reynolds numbers, the mean flow will stay distorted for a long distance downstream.

Figure 1 shows the variation of k_r and k_i with R as the independent variable and ω as a parameter for the least stable mode (one for which $|k_r|$ is minimum) when $n = 1$. It is clear from this figure that for a fixed frequency the mode becomes less stable and has a larger wavelength at higher Reynolds numbers. Moreover, the asymptotic nature of the curves indicates that, though the mode becomes progressively less stable at higher Reynolds numbers, it is unlikely that k_r will ever become positive, indicating instability. Increasing the frequency at a fixed R causes only a slight increase in modal stability but a significant decrease in wavelength. If the frequency is doubled at a fixed Reynolds number, the wavelength reduces by a factor of about 2 so that the phase velocity is nearly constant. Similar conclusions were obtained for the least stable modes for $n = 0, 2$ and 3.

A comparison of the eigenvalues for the least stable mode for four different values of the azimuthal wavenumber n is provided by figure 2. Though the values of $k_r R$ and $k_i R$ do change with different combinations of ω and R for which ωR is constant, these changes are too small to be represented graphically for the cases investigated. For a constant value of ωR, these cases provided a ten-fold variation of ω and R. The lower portion of this figure helps to prove an earlier statement that $n = 1$ possesses the least stable of all modes, including the one for which $n = 0$. The possibility of such behaviour was speculated by Betchov &

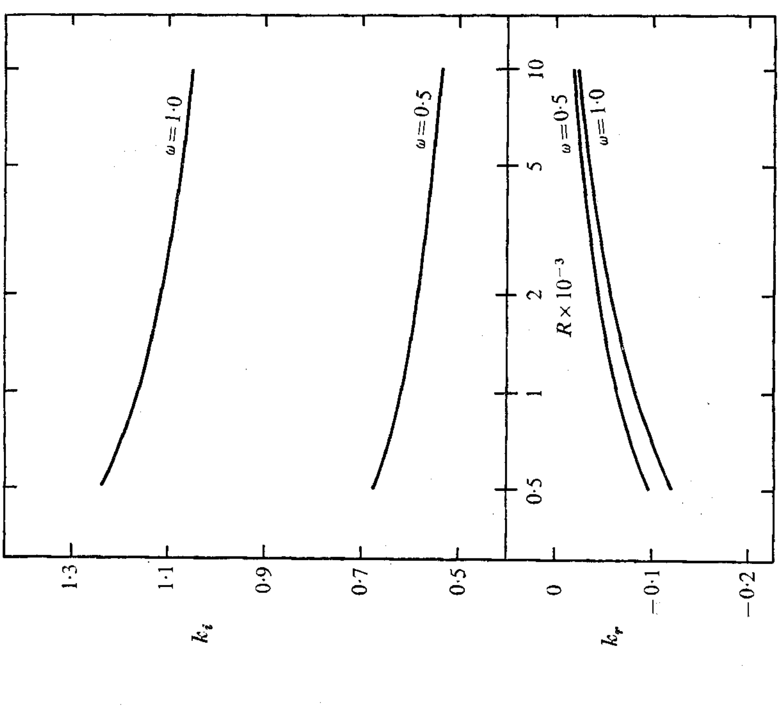

FIGURE 1. Variation of k_r and k_i with R for the least stable mode for $n = 1$.

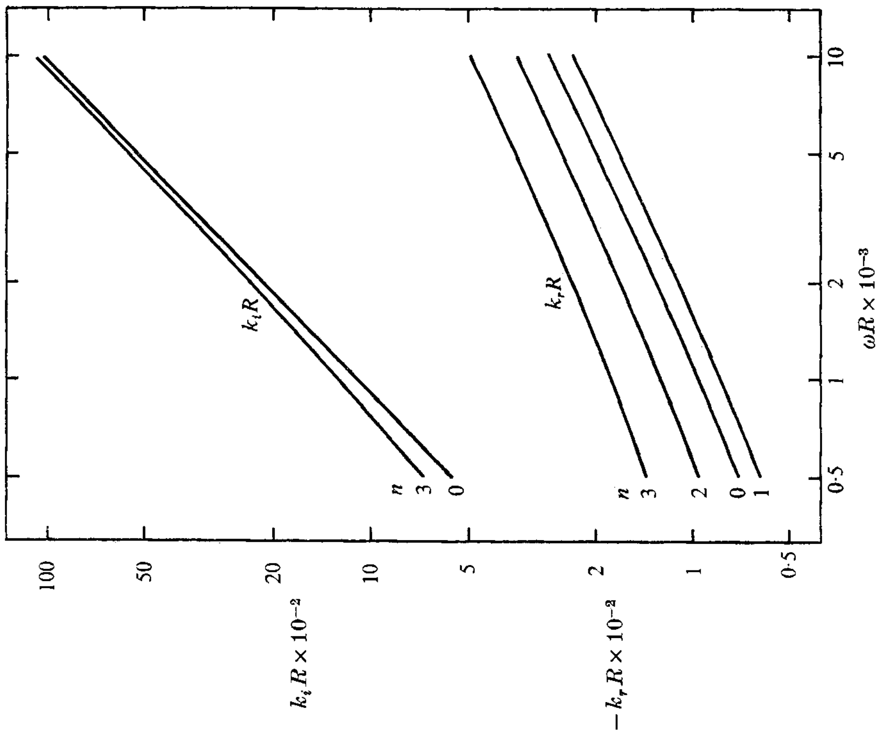

FIGURE 2. Variation of $k_r R$ and $k_i R$ with ωR for the least stable mode for several values of n.

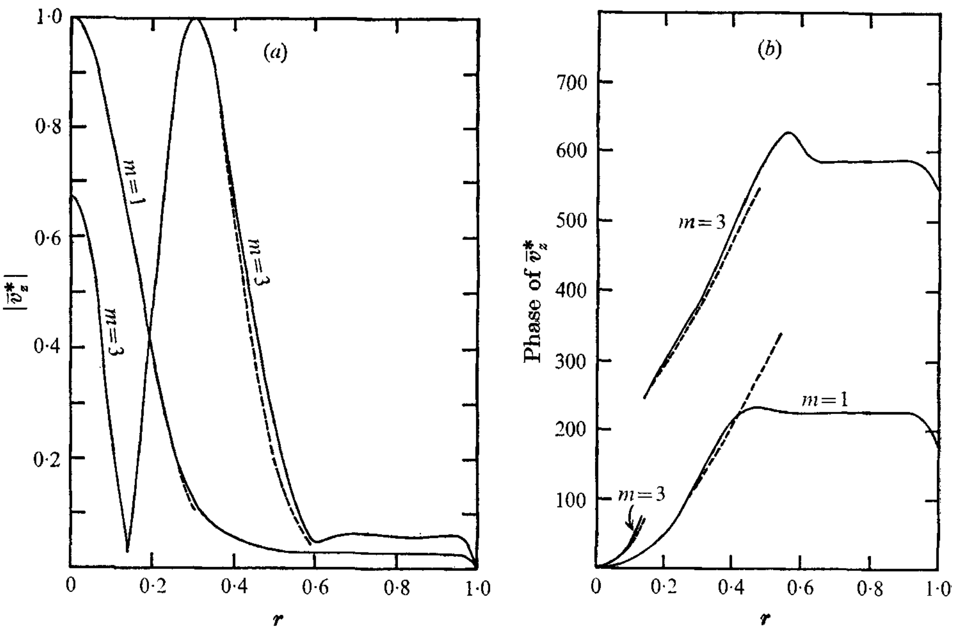

FIGURE 3. Comparisons with previous work (Gill 1965) for an axisymmetric disturbance. $R = 4000$, $\omega = 0{\cdot}975$. —, present work; – – –, Gill (1965). (*a*) $|\bar{v}_z^*|$ *vs* r. (*b*) Phase of $\bar{v}_z^*$ *vs*. r.

Criminale (1967, p. 229) on the basis that in a pipe flow the production of vorticity vanishes identically for $n = 0$ but not for higher values of n. This implies, however, that $n = 0$ should be the most stable mode of all, which is not the case. Since $n = 0$ allows for non-zero values of $\bar{v}_z(0)$ and $\bar{p}(0)$ while for $n = 2, 3, \ldots$ all the eigenfunctions are required to vanish at $r = 0$, it appears that the non-zero boundary conditions at $r = 0$ for $n = 0$ have a more significant effect than merely the non-zero production of vorticity for $n = 2, 3, \ldots$. Continuing this argument further, both the non-zero boundary conditions for $\bar{v}_r$ and $\bar{v}_\theta$ at $r = 0$ and the non-zero production of vorticity help to make the least stable mode of all occur when $n = 1$.

As far as k_i is concerned, there is no such ambiguity; the value of $k_i R$ increases with higher values of n at a constant ωR. The curves for $n = 1$ and $n = 2$ lie between those for $n = 0$ and $n = 3$, and have been omitted here for clarity. It is interesting to note that the curves in figure 2 are almost linear, suggesting thereby that a simple power law may be used to relate $k_r R$ and $k_i R$ with ωR. A relation connecting the power law constants with the parameter n is, however, rather complicated.

Figures 3(*a*) and (*b*) compare some results of the present investigation for an axisymmetric disturbance with those of Gill (1965) for a Reynolds number of 4000 and a dimensionless frequency of 0·975, so that $\omega R = 3900$. The axial velocity eigenfunction $\bar{v}_z(r)$ is represented on these figures in terms of its amplitude and phase for two modes $m = 1$ and $m = 3$.† The amplitude $|\bar{v}_z(r)|$ was normalized

† For the significance of m see Gill (1965).

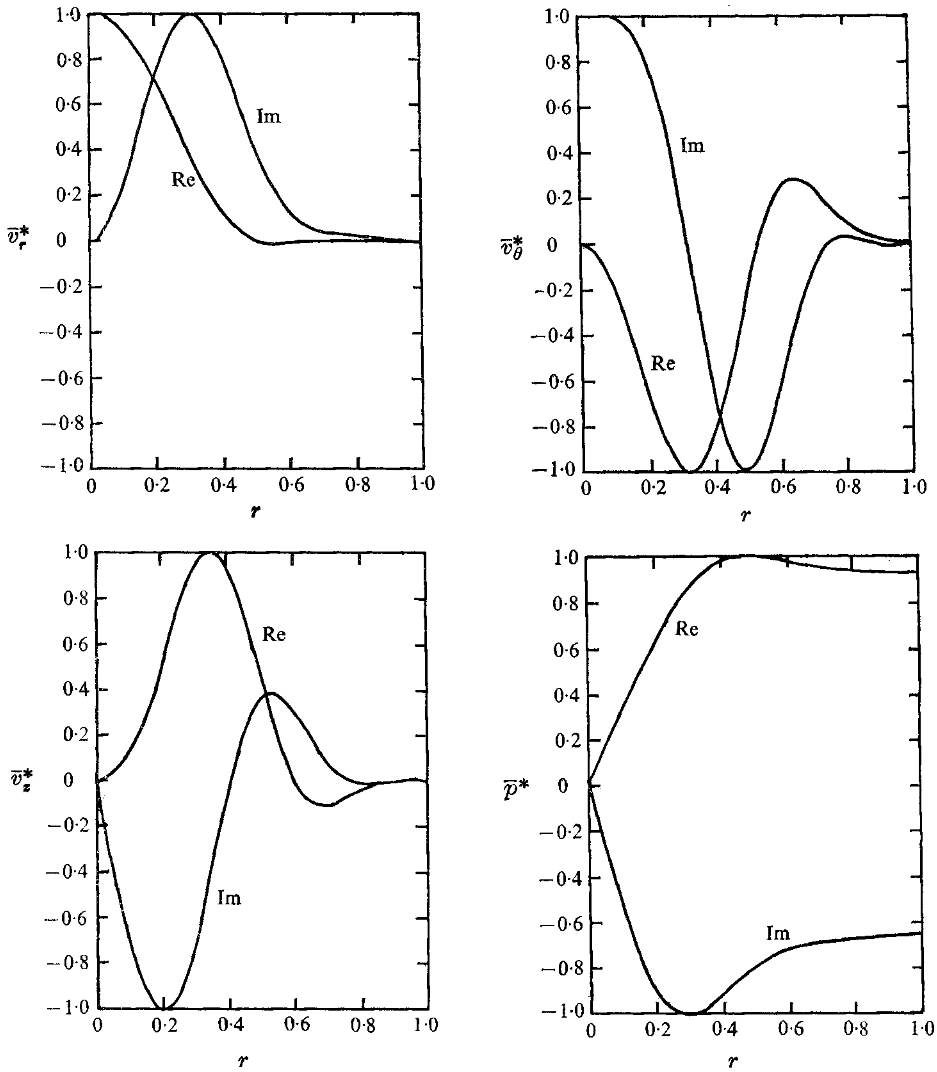

FIGURE 4. Eigenfunctions for the least stable mode for $n = 1$ ($R = 5000$, $\omega = 0{\cdot}1$).

with respect to its maximum value before plotting. Gill's results are shown by the dashed lines in these figures. For small r, the agreement is excellent; in fact, the full and dashed lines are indistinguishable. For slightly larger r, any small differences can be attributed to the errors inherent in taking values from another paper. There is an abrupt end to the dashed lines on these figures since Gill could not find the corresponding values beyond a certain radius owing to his assumption that r is close to zero. This also accounts for the large error in his results for the phase angle of $\bar{v}_z(r)$ when $m = 1$ and $r \gtrsim 0{\cdot}4$. The present technique is, of course, valid over the whole region $0 \leqslant r \leqslant 1$.

Eigenfunctions for the least stable mode for various values of n are presented in figures 4, 5 and 6 in terms of their real and imaginary parts normalized with

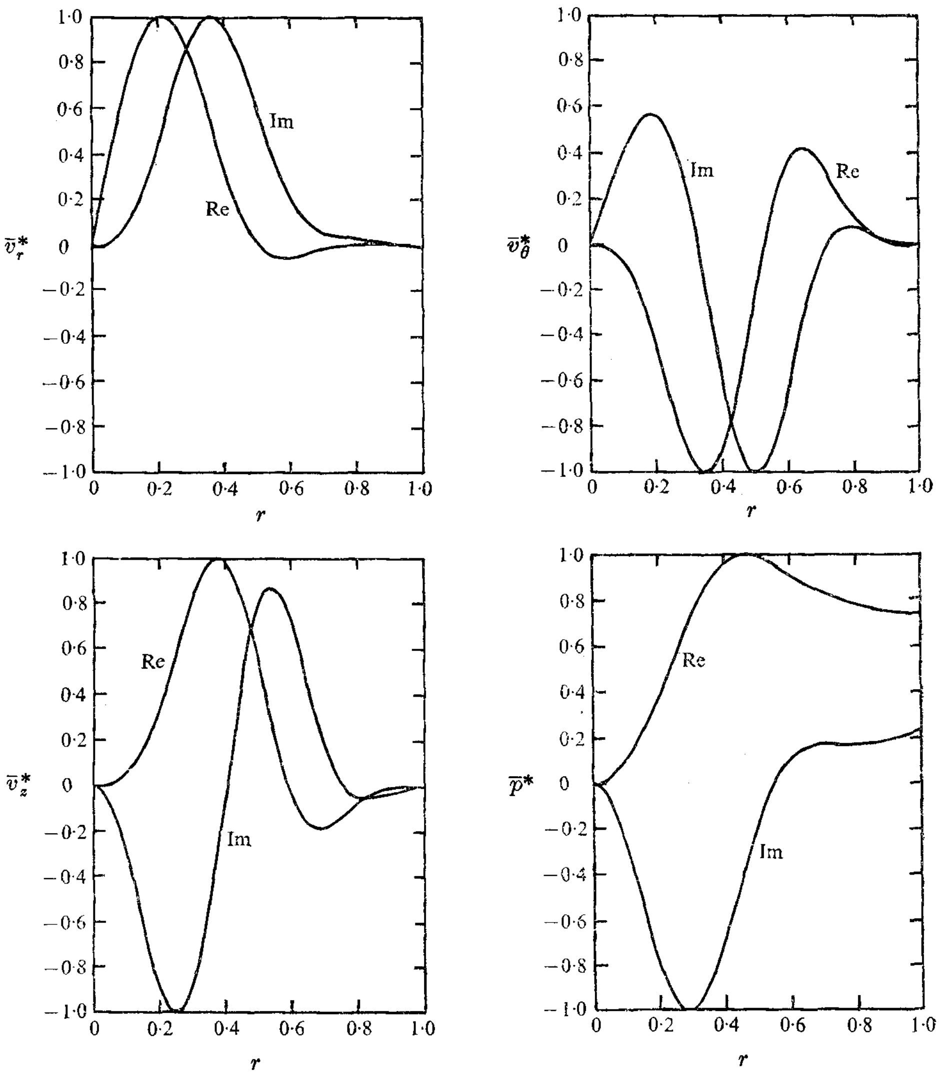

FIGURE 5. Eigenfunctions for the least stable mode for $n = 2$ ($R = 5000$, $\omega = 0{\cdot}1$).

respect to their individual maximum magnitudes; these normalized counterparts of $\bar{v}_r$, $\bar{v}_\theta$, $\bar{v}_z$ and $\bar{p}$ are represented by $\bar{v}_r^*$, $\bar{v}_\theta^*$, $\bar{v}_z^*$ and $\bar{p}^*$. These figures help to point out the effect of varying the azimuthal wavenumber n on the mode shape of the least stable mode. As n varies from 1 to 3, some of the eigenfunctions are affected more than the others. Owing perhaps to the non-zero values of $\bar{v}_r(0)$ and $\bar{v}_\theta(0)$ for $n = 1$, the mode shape changes considerably as n varies from 1 to 2, the change from 2 to 3 in the value of n being rather insignificant for the mode shape. It was also found that, for a given set of parameters, the mode shape becomes more oscillatory as the mode becomes more stable.

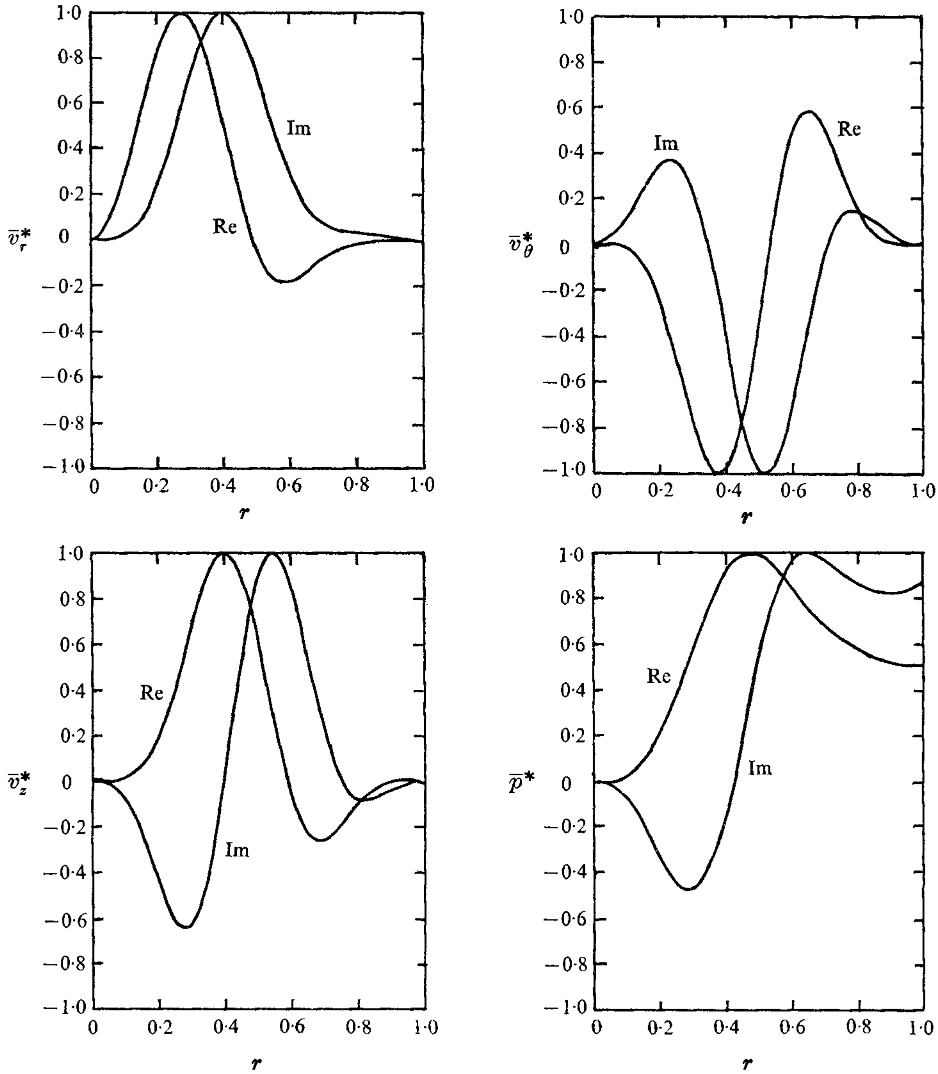

FIGURE 6. Eigenfunctions for the least stable mode for $n = 3$ ($R = 5000$, $\omega = 0{\cdot}1$).

6. Computational methods and accuracy

The series solution was generally applied in the region $0 \leqslant r \leqslant 0{\cdot}1$, and from 11 to 15 terms were required for convergence, the larger number of terms corresponding to the product ωR approaching 10000. The convergence criterion was that the ratio of the last term retained to the partial sum up to a term preceding the last be less than 10^{-12}. For the numerical integration of the differential equations over $0{\cdot}1 \leqslant r \leqslant 1$ both Runge–Kutta and predictor–corrector methods of order $L = 4$ were used in order to provide a check; step sizes varied from 0·01 to 0·001. Following Ralston (1965, ch. 5) and Acton (1970, ch. 5) the accuracy was checked by solving some cases twice, once for a step size of h and then for $\frac{1}{2}h$.

Step size	k (as calculated)	k (corrected using (6.1))
0·0100	$-0{\cdot}02083552035+0{\cdot}51998929928i$	—
0·0050	$-0{\cdot}02083549559+0{\cdot}51998925465i$	$-0{\cdot}02083549394+0{\cdot}51998925168i$
0·0025	$-0{\cdot}02083549399+0{\cdot}51998925191i$	$-0{\cdot}02083549388+0{\cdot}51998925173i$
0·0010	$-0{\cdot}02083549388+0{\cdot}51998925173i$	—

TABLE 2. Values of k for various values of the step size ($R = 10000$, $\omega = 0{\cdot}5$, $n = 0$)

Step size	k (as calculated)	k (corrected using (6.1))
0·0100	$-0{\cdot}01722848619+0{\cdot}53525172047i$	—
0·0050	$-0{\cdot}01722769846+0{\cdot}53525112413i$	$-0{\cdot}01722764595+0{\cdot}53525108437i$
0·0025	$-0{\cdot}01722764735+0{\cdot}53525108570i$	$-0{\cdot}01722764394+0{\cdot}53525108314i$
0·0010	$-0{\cdot}01722764397+0{\cdot}53525108317i$	—

TABLE 3. Values of k for various values of the step size ($R = 10000$, $\omega = 0{\cdot}5$, $n = 1$)

According to Collatz (1966, p. 51) the error in the calculations with the smaller step size is approximately one part in (2^L-1) of the difference between the results of the two calculations; thus the corrected value y_c is given by

$$y_c = y_{\frac{1}{2}h} - (y_h - y_{\frac{1}{2}h})/(2^L-1). \tag{6.1}$$

Some representative examples of the results for k are shown in tables 2 and 3 for $n = 0$ and 1.

These results indicate that the largest step size yields at least 5 significant figures, while the smallest step size gives accuracy of 9 or more significant figures, as confirmed by the corrected values given by (6.1). This large number of significant figures was made possible by using double-precision arithmetic on a Univac 1108 computer, which carried 18 digits in each operation. Slightly less accuracy was obtained for $n = 1$ than for $n = 0$ since at least twice as many calculations were required for $n = 1$. Any numerical instability (encountered only twice during the entire investigation) always disappeared with the use of a smaller step size, indicating thereby that round-off error was insignificant.

7. Conclusions

It was found that ωR governs, in an approximate manner, the values of kR. This observation makes it possible to vary the product ωR rather than ω and R separately. The effect of Reynolds number, frequency and the azimuthal wave-number n on the relative stability and wavelength of a mode can be summarized as follows.

(i) For fixed values of ω and R, the least stable mode of all corresponds to $n = 1$. In order of increasing stability, the sequence for the values of n is $1, 0, 2, 3, \ldots$.

(ii) For constant ω and R, the least stable mode (one for which $|k_r|$ is minimum) has the largest wavelength when $n = 0$. The wavelength decreases only slightly as n takes on higher integer values, the effect at high values of ωR being negligibly small.

(iii) For a fixed n and ω the least stable mode becomes less stable and has a larger wavelength as the Reynolds number is increased.

(iv) The effect of frequency for constant n and R is similar to that in (iii) above except that the wavelength is almost doubled when the frequency is reduced by a factor of two. Thus the modal phase velocity remains almost constant.

(v) For a fixed value of n, the least stable mode tends to be neutrally stable that is $k_r \to 0$, as the frequency of the disturbance is decreased and the Reynolds number of flow is increased indefinitely.

(vi) For constant n, ω and R, the mode shape becomes more oscillatory as the mode becomes more stable, that is, as $|k_r|$ increases.

Theoretical results were obtained for Reynolds numbers up to 10000. For these Reynolds numbers it was found that the pipe Poiseuille flow is spatially stable to all infinitesimal disturbances. It may be inferred from the asymptotic behaviour of the eigenvalue trajectory for the least stable mode (figure 1) that instability does not occur even at higher Reynolds numbers. For the axisymmetric disturbance, the results are in agreement with previous theoretical and experimental results (Gill 1965; Leite 1959). For the non-axisymmetric disturbance, this is the first theoretical study of spatial stability. The experimentally observed instability for a non-axisymmetric disturbance (Lessen, Fox, Bhat & Liu 1964; Fox, Lessen & Bhat 1968) is in contradiction with the present conclusion, but it is believed that the amplitude of the disturbance was finite in these experimental studies, as was true for the axisymmetric instability observed experimentally by Kuethe (1956).

This research was supported in part by the National Science Foundation Grants GK-2542 and GK-31042; it is based on the dissertation submitted by V.K.G. to Carnegie-Mellon University in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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